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ON HIGHER BORN APPROXIMATIONS IN POTENTIAL SCATTERING OF FAST ELECTRONS BY ATOMIC NUCLEI IN A STATIC FIELD

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ABSTRACT. Some of the conclusions on higher Born approximation following the works of Sauter, Sexl and Distel as presented in the literature are incorrect. Mott, using the Dirac's second-order relativistic equation and taking the exact solution, has obtained a second-order correction term, which is different from the result of Urban. Urban's result is the same as that of Sexl. Both results are incorrect as they are not consistent expansions in powers of αZ, where α and Z are the fine-structure constant and the atomic number respectively. Using the matrix-formalism Dalitz has recently obtained a 2nd order term in the scattering cross section for the Dirac particle, pointing out the errors in the development of the former writers. In this paper, the second-order approximation in the elastic scattering of fast electrons by atom has been carried out using the hypercomplex notation. The first appoximation has been checked by this method by Sauter. The method used here, is based on a consistent expansion in powers of αZ. The series actually obtained for the cross section is given by,

$$(1-\beta^2 \sin^2 \theta/2)+\pi\alpha Z\beta.\ \sin^2 \theta/2\ (\tfrac{3}{2}-2 \sin^2 \theta/2)+,\ \ldots$$

multiplied by the Rutherford scattering formula.

INTRODUCTION

The method of higher Born approximation in the discussion of the scattering problems consists in the calculation of the series-expansion of the scattering-amplitudes in powers of the interaction potential. The Born approximation has been developed in a variety of forms and has been applied to different types of problems. But the calculations have not been carried out correctly beyond the first approximation. Sauter (1933*b*), using the time-dependent perturbation method has obtained the second-order correction which contains an error in the development (Dalitz 1951). Urban, (1943) following him proceeded to calculate the third approximation, but was unable to calculate all the terms of the series and his method is wrong. The method of Sommerfeld and Mao (1935) gives an unsurmountable difficulty in finding out the higher-order terms. Dalitz (1951) has found the second-order correction term, using the matrix-formalism of Dyson and Feynman. Sauter, (1933*a*) using the hypercomplex notation has correctly formulated the first Born approximation. In this paper, this formalism has been extended to calculate the second-order correction term in the scattering of fast Dirac electron by the potential, $V(r) = Ze.e^{-r/a}/r$, which is of some interest as a representation of the screened atomic field.

The wave function Ψ of a particle in a static field $V(r)$ is expanded in a series $\Psi = \psi_0 + \psi_1 + \psi_2 + \ldots\ldots\ldots$; where ψ_0 represents the incident wave undisturbed by the field, and ψ_1, ψ_2...consist only of outgoing waves at infinity. The latter functions are to be found from a recurrence formula.

In this paper, actually the function ψ_2 has been calculated. It is seen that the evaluation of ψ_2 depends on the evaluation of the integrals (see Eq (9) L_1, L_2, L_3, L_4 of which $L_1 \to \infty$ in the limit $a \to \infty$. Others are finite. It can be easily observed from the wellknown formula of the current density, that the contribution to the scattering cross section is only due to the imaginary part of the integral, L_1. Thus the difficulty in handling with L_1 due to its infinite-character has been avoided.

THE SCATTERING OF A DIRAC PARTICLE

The relativistic Dirac's equation may be written as

$$\left[\sum_{\nu=1}^{3} \frac{\partial}{\partial x_\nu} - \gamma_4 \frac{E-V}{\hbar c} + \frac{mc}{\hbar} \right] \Psi = 0 \tag{1}$$

where V is the general potential function and $\gamma_K = i\beta\alpha_K$; $\gamma_4 = \beta$ ann $\hbar = \frac{h}{2\pi}$

Now let Ψ be expanded as $\Psi = \psi_0 + \psi_1 + \psi_2 + \ldots$ (2)

where $\psi_0 = a\, e^{\frac{2\pi i}{h}(\vec{p}.\vec{r})}$, the incident undisturbed wave, ... (3)
and ψ_1, ψ_2,...are outgoing waves at infinity. Putting the value of Ψ from (2) in (1) and collecting the terms of the same order one gets the recurrence relation,

$$\left(\sum_\nu \gamma_\nu \frac{\partial}{\partial x_\nu} - \gamma_4 \frac{E}{\hbar c} + \frac{mc}{\hbar} \right) \psi_n = -\gamma_4 \frac{V}{\hbar c} \psi_{n-1} \tag{4}$$

Operating the equation (4) from the left by

$$\left(\sum_\nu \gamma_\nu \frac{\partial}{\partial x_\nu} - \gamma_4 \frac{E}{\hbar c} - \frac{mc}{\hbar} \right)$$

one gets,

$$\left[\Delta + \frac{r^2}{\hbar^2} \right] \psi_n = -\left(\sum_\nu \gamma_\nu \frac{\partial}{\partial x_\nu} - \gamma_4 \frac{E}{\hbar c} - \frac{mc}{\hbar} \right) \gamma_4 \frac{V}{\hbar c} \psi_{n-1} \tag{4a}$$

where Δ is the Laplacian operator.

The solution of this is given by

$$\psi_n(\vec{R}) = \frac{1}{4\pi\hbar c} \int \frac{e^{i\frac{2\pi p}{h}(\vec{R}-\vec{r})}}{|\vec{R}-\vec{r}|} \cdot \left\{ \sum_\nu \gamma_\nu \frac{\partial}{\partial x_\nu} - \gamma_4 \frac{E}{\hbar c} - \frac{mc}{\hbar} \right\} \gamma_4 V(r).\psi_{n-1}(r) dr_r \ \ldots \tag{5}$$

where $\vec{R}$ stands for the vector OP and r for OP'. The point P has the coordinate (X_1, X_2, X_3) and P' has (x_1, x_2, x_3). OP is along the direction of observation, and P' the integration point (figure 1). O is the position of the

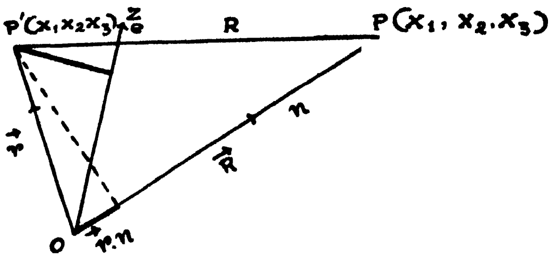

FIG. 1

scatterer. The distance between P and P' is denoted by R.

With regard to the source and sink point, one can write, $\frac{\partial}{\partial x_\nu} f(\bar{R} - \bar{r}) = - \frac{\partial}{\partial X_\nu} f(\vec{R} - \vec{r})$, and with the help of this, the equation (5) may be written as

$$\psi_n(\vec{R}) = \frac{1}{4\pi\hbar c}\left\{\sum_\nu \gamma_\nu \frac{\partial}{\partial X_\nu} - \gamma_4 \frac{E}{\hbar c} - \frac{mc}{\hbar}\right\}\int \frac{e^{i\frac{2\pi p}{h}(\vec{R}-\vec{r})}}{|\vec{R}-\vec{r}|} \cdot V(\gamma)\, \psi_{n-1}(r)\, d\tau_r \quad \ldots \quad (6)$$

We have from (figure 1), $|\vec{R}-\vec{r}| = R - r.n$ where n is the unit vector in the direction of observation, and replace $|\vec{R}-\vec{r}|$ by R since R is very large.

Sauter (1933) has calculated the value of ψ_1 and he has found, $\psi_1 \longrightarrow$

$$\frac{1}{4\pi\hbar^2 c^2} \cdot \frac{e^{2\pi i p/h}}{R}\left\{-2E + icp(n - e, \vec{\gamma}) \cdot \gamma_4\right\} a. \int V. e^{2\pi i p/h}(e - n_1\vec{\gamma})\, d\tau_r \quad \ldots \quad (7)$$

where e is the unit vector along the direction of the incident wave.

Noting that the amplitude a of equation (3) satisfies the following equation,

$$\left\{icp(e.\vec{\gamma}) - \gamma_4 E + mc^2\right\} a = 0$$

we can turn the above equation (6) in the form (7). Now let us calculate ψ_2 from the recurrence relation (6). Thus

$$\psi_2(\vec{R}) = \frac{1}{4\pi\hbar c}\left\{\sum_\nu \gamma_\nu \frac{\partial}{\partial X_\nu} - \gamma_4 \frac{E}{\hbar c} - \frac{mc}{\hbar}\right\}\gamma_4 . \int \frac{e^{2\pi i p/h(\vec{R}-\vec{r})}}{|\vec{R}-\vec{r}|} \cdot V(r).\psi_1(r)\, d\tau_r$$

Substituting the value of $\psi_1(r)$ from equation (7) we have

$$\psi_2(\vec{R}) = \frac{1}{\hbar 4\pi c}\Big\} \sum_\nu \gamma_\nu \frac{\partial}{\partial X_\nu} - \gamma_\nu \frac{E}{\hbar c} - \frac{mc}{\hbar}\Big\} \cdot \frac{1}{4\pi\hbar^2 c^2}$$

$$\int \frac{e^{2\pi i p/h(\vec{R}-\vec{r})}}{|\vec{R}-\vec{r}|} \cdot V(r)\left[\frac{e^{2\pi i p/h}}{r}\left\{-2E + icp(n_1 - e, \vec{\gamma}) . \gamma_4\right\} a \int V(r') e^{2\pi i p/h\,(e - n_1 \vec{r}')} d\tau'\right] d\tau_r$$

where, e, n, and n_1 are the unit vectors in the direction of the vector $\vec{r}$ (the direction of which has been taken as the Z-axis), in the direction of the vector $\vec{R}$ (the direction of observation) and, in the direction of the vector $\vec{r}$ (a variable vector), respectively (vide figure 2). The angles θ, θ' and ω are

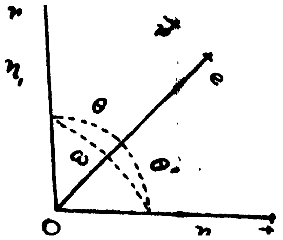

FIG. 2

respectively, the angles between the vectors $\widehat{(e, n_1)}$, $\widehat{(e, n)}$ and $\widehat{(n, n_1)}$, also we replace $\cos\omega$ by $\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\phi$ where ϕ is the angle between the planes containing (e, n) and (e, n_1) vectors.

We set further the following abbreviations :

$$k = p/\hbar \ ;$$

$$V(r) = \frac{ZeE'}{r} \cdot e^{-r/a} ; \text{ and } b = 1 + \frac{1}{2a^2k^2}$$

Using the above abbreviations, we proceed to calculated ψ_2. Thus performing the r'-integration in the square bracket of the last expression, and changing the vector $(n_1 - e)$ into its polar forms, we can write,

$$\psi_2(\vec{R}) = \left(\frac{1}{4\pi\hbar^2c^2}\right)^2 \frac{4\pi ZeE'}{R} \cdot e^{2\pi i p/h\, R} \left(icp(n.\vec{\gamma}) \gamma_4 E - mc^2\right)\gamma_4 \Big\{ L_1 + icp(L_2 + L_3 - L_4)\Big\}$$

where,

$$L_1 = \int \frac{e^{ik(1-\cos\omega)r}}{r} V(r) \frac{}{4k^2 \sin^2\theta/2 + 1/a^2} (-2E)\, a.d\tau_r$$

$$L_2 = \int \frac{e^{ik(1-\cos\omega)r}}{r} V(r) \frac{\sin\theta\cos\phi}{4k^2 \sin^2\theta/2 + 1/a^2} \cdot \gamma_1\gamma_4 . a . d\tau_r$$

$$L_3 = \int \frac{e^{ik(1-\cos\omega)r}}{r} V(r) \frac{\sin\theta . \sin\phi}{4k^2 \sin^2\theta/2 + 1/a^2} \gamma_2\gamma_4 . a . d\tau_r \qquad \ldots \quad (9)$$

$$L_4 = \int \frac{e^{ik(1-\cos\omega)r}}{r} V(r) \frac{(1-\cos\theta)}{4k^2 \sin^2\theta/2 + 1/a^2} \cdot \gamma_3\gamma_4 . a . d\tau_r$$

Now if by J we denote the number of particles scattered through unit solid angle per unit time, then J is given by a well known formula,

$$J = ic\{\Psi^* \gamma_4 (\vec{n.\gamma})\Psi\}$$

$$= ic\{\psi_1{}^*\gamma_1(\vec{n\ \gamma})\psi_1\} + ic[\{(\psi_1{}^*\gamma_1(\vec{n\gamma}).\psi_2\} - \text{its conjugate complex}]$$ which is upto second-order terms. ... (10)

$$J = J_1 + J_2 = J_1 + J_2{}^{(1)} + J_2{}^{(2)}$$

Now $$J_2{}^{(1)} = ic\left\{\psi_1{}^*\gamma_1(\vec{n.\gamma})\psi_2\right\}$$

$$= ic\ \frac{1}{4\pi\hbar^2 c^2}\cdot\frac{e^{-2rip/\hbar R}}{R}\ .a^*\left\{-2E + icp(\vec{n-e,\ \gamma})\gamma_4\right\}.\ \frac{4\pi ZeE'}{2k^2(b-\cos\theta'}\ .\gamma_1.(\vec{n\gamma}).$$

$$.\left(\frac{1}{4\pi\hbar^2 c^2}\right)^2.\frac{4\pi ZeE'}{R}.\ e^{2.ip/\hbar R}\left(icp(\vec{n\gamma}) - \gamma_1 E - mc^2\right)\gamma_1.\Big]L_1 + icp(L_2 + L_3 - L_4)\Big].$$

$$= ic.\ \frac{1}{(4\pi\hbar^2 c^2)^3}\cdot\frac{(4\pi ZeE')^2}{R^2.2k^2}\cdot\frac{1}{(b-\cos\theta')}\ .a^*\left\{-2E + icp(\vec{n-e,\gamma}).\gamma_1\right\}.\gamma_4(\vec{n\ \gamma})$$

$$\times\left(icp(\vec{n\ \gamma}) - \gamma_1 E - mc^2\right).\gamma_4\left\{L_1 + icp(L_2 + L_3 - L_4)\right\} \quad ..\quad (11)$$

The differential cross section, after averaging over the initial electron states and the summing over the final states, is obtained from the expression

$$\tfrac{1}{2}\ \text{spur}\left[\frac{ic}{(4\pi\hbar^2 c^2)^2}\cdot\frac{(4\pi.ZcE')^2}{R^2.2k^2}\cdot\frac{1}{(b-\cos\theta')}\ .a^*\left\{-2E + icp(\vec{n-e,\gamma})\gamma_1\right\}\gamma_1\left(\vec{n.\gamma}\right)\right.$$

$$.\left(icp(\vec{n.\gamma}) - \gamma_1 E - mc^2\right).\gamma_1\left\{L_1 + icp(L_2 + L_3 - L_4)\right\},$$

where the values of the intigrals L_1, L_2, L_3, and L_4 are evaluated in the appendix. It is easy to see that the contribution to the scattering cross section is due to the integral term L_1. Hence in calculating the spur value, of the above expression due to the term L_1 we need only consider the following,

$$\tfrac{1}{2}\ \text{spur}.a^*a\left\{-2E + icp(\vec{n-e,\ \gamma})\gamma_4\right\}\gamma_4(\vec{n\gamma})\left\{icp(\vec{n\gamma}) \cdot \gamma_4 E - mc^2\right\}(-\gamma_3).\frac{|E| + H_0}{2E}$$

where $H_0 = (\vec{\alpha p_0}) + \beta.mc$, $\vec{p_0}$ is the momentum vector in the initial direction, which gives,

$$\frac{J_0.E.c^2p^2}{c^2p} \qquad ...\quad (12)$$

where $J_0{}^* \longrightarrow v(\bar{a}.\gamma_4 a)$

Now, utilising the values obtained in equations (11) and (12) the resultant cross section becomes the following (in this θ' has been changed to θ as usual :

$$J_2 = 2\left(\frac{1}{4\pi\hbar^2 c^2}\right)^3 \cdot \frac{1}{R^2} \cdot \frac{(4\pi Z e E')^2}{2k^2(b - \cos\theta)} \cdot J_0 \cdot \frac{E}{v c^2} v^2 c^2 (1 - 2\cos\theta) . v^2 . \frac{2\pi^2}{2k^3}$$

Changing E' by $-e$ for electron, and when $b \longrightarrow 1$ for the bare nucleus

$$J_2 = \frac{J_0}{R^2}\left(\frac{Ze^2}{2mv^2}\right)^2 (1 - \beta^2)\left[\pi \frac{Ze^2}{\hbar c} . \beta \; \frac{1 - 2\cos\theta}{1 - \cos\theta}\right]$$

where $\beta = v/c$

The value of J_1, has been checked by this method by Sauter (1933*a*) and has been found to be

$$J_1 = \frac{J_0}{R^2}\left(\frac{Ze^2}{2mv^2}\right)^2 (1 - \beta^2) \operatorname{cosec}^4 \theta/2 (1 - \beta^2 \sin^2 \theta/2)$$

Thus upto the second-order correction term, we get

$$J = \frac{J_0}{R^2}\left(\frac{Ze^2}{2mv^2}\right)^2 (1 - \beta^2) \operatorname{cosec}^4 \theta/2 \left[1 - \beta^2 \sin^2 \theta/2 + \pi.\alpha.\beta. \sin^2\theta/2\left(\frac{1 - 2\cos\theta}{2}\right) + ..\right]$$

where $\alpha = Ze^2/\hbar c$

If R stands for the ratio of the scattering to the Rutherford scattering, then upto second-order approximation, R becomes

$$R = (1 - \beta^2 \sin^2\theta/2) - \pi Z.\alpha_f.\beta \sin^2\theta/2 \; (\tfrac{3}{2} - 2 \sin^2\theta/2)$$

where α_f stands for the fine-structure constant.

When still higher terms are calculated, this is consistent expansion in powers of $Z.\alpha_f$.

CONCLUSION

The correction term of order Ze^2 (relative to the first order) found here is $\pi. \frac{Ze^2}{\hbar c} .\beta. \sin^2\theta/2\left(3/2 - 2 \sin^2\theta/2\right)$ which is not in agreement with that obtained by Mott (1929). His correction term is $\pi \cdot \frac{Ze^2}{\hbar c} \sin\frac{\theta}{2} \cos\frac{\theta}{2}$. Urban (1942) obtained the correction terms as $\pi.\frac{Ze^2}{\hbar c} \sin\frac{\theta}{2}$ as that of Sexl's (1933). But their results originate from errors pointed out by Dalitz (1951). Dalitz's correction term comes out to be $\pi.\frac{Ze^2}{c\hbar}. \sin\frac{\theta}{2} (1 - \sin\theta/2)$, which is at variance form the result obtained here. The advantage of the method used here is that it is quite elegant and lucid.

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APPENDIX

The evaluation of the integrals occurring in (9)

In evaluating the integrals in equation (9), we first consider the integral L_1 which after the completion of r-integration gives the ϕ-integration in the form,

$$\int_0^{2\pi} \frac{d\phi}{a - b\cos\phi}\,; \quad a > b$$

in which $a = \lambda - \cos\theta\cos\theta'$, $b = \sin\theta\sin\theta'$: $\lambda \longrightarrow 1$: the θ-integration may be effected by transforming the integral by the substitution $b - \cos\theta = Z$ to the wellknown form

$$\int \frac{dZ}{Z\sqrt{AZ^2 + BZ + C}}$$

the values of A, B, C can be easily found out. Thus,

$$L_1 = \frac{2\pi}{ik}\,\frac{E}{k^2\; b - \cos\theta'}\;\frac{\left\{\dfrac{M}{2L} - \dfrac{1}{1+b}\right\} + \sqrt{\left(\dfrac{M}{2L} - \dfrac{1}{1+b}\right)^2 + \dfrac{1}{L} - \dfrac{M^2}{4L^2}}}{\left\{\dfrac{M}{2L} + \dfrac{1}{1-b}\right\} + \sqrt{\left(\dfrac{M}{2L} + \dfrac{1}{1-b}\right)^2 + \dfrac{1}{L} - \dfrac{M_2}{4L^2}}}$$

where,

$$L = b^2 - 2\lambda b\cos\theta' + (\lambda^2 + \cos^2\theta' - 1)$$

$$M = 2b - 2\lambda\cos\theta'$$

$$\lambda = 1 - \frac{1}{iak}\,; \quad b = 1 + \frac{1}{2a^2k^2}$$

For the integrals L_2 and L_3 we see it convenient to take help of the contour integration. Combining L_2 and L_3 we have for the ϕ-integration,

$$\int_0^{2\pi} \frac{e^{i\varphi}}{a - b\cos\phi}\,d\phi\,; \quad a > b$$

and a, b stand for the same values as in L_1 by changing $Z = e^{i\varphi}$ this reduces to

$$\frac{2}{i}\int_{\Gamma} \frac{Z\,dZ}{2aZ - b - bZ^2} = \frac{2}{i}\cdot 2\pi i \text{ (sum of the residues at the poles) where } \Gamma \text{ is}$$

the unit-circle. The evaluation of L_1 may be performed in the like manner. In this evaluation we have neglected the variation of the quantity

$$\left\{\frac{1}{a^2k^2} + \frac{2}{iak}(\cos\theta\cos\theta' - 1)\right\} \text{ in } \sqrt{\left[(\cos\theta - \cos\theta')^2 - \left\{\frac{1}{a^2k^2} + \frac{2}{iak}(\cos\theta\cos\theta' - 1)\right\}\right]}$$

in the limit $a \to \infty$. Thus we have

$$\lim_{a\to\infty} L_3 = \frac{\pi i}{3}\{\pi i + 2\log\tan\theta/2\}$$

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